Y.V. Imp

M.U.

M.Sc. 93, 94, 95

Q.No.→ State and Prove the Principle of uniform boundedness.

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M.Sc. 97, Q.No.→ State and establish uniform boundness Principle.

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M.Sc. 90, Q.No.→ Let B be a Banach Space and E a normed linear space. If $\{T_\alpha\}$ is a non-empty set of Continuous linear transformation of B into E with the property that $\sup_\alpha \|T_\alpha x\| < \infty$ for all $x \in B$

then Prove that $\sup\limits_{\alpha} \|T_\alpha\| < \infty$.

Or, Q No.> State and Prove Banach Steinhaus theorem.

Note:- जो (Q No. Misc. 94) में पूछा गया है, उसमें सिर्फ Proof देना।

Ans.> Statement:- Let B be a Banach Space and E a normed linear Space. If $\{T_\alpha\}$ is a non-empty set of Continuous linear transformations of B into E with the Property that,

$$\sup_{\alpha} \|T_\alpha x\| < \infty \text{ for all } x \in B,$$

then, $$\sup_{\alpha} \|T_\alpha\| < \infty.$$

Proof:- For each Positive integer n, the set

$$F_n = \{x \in B : \|T_\alpha x\| \leq n \text{ for all } \alpha\}$$
$$= \{x \in B : T_\alpha x \in S_n[0] \text{ for all } \alpha\}$$
$$= \{x \in B : x \in T_\alpha^{-1}(S_n[0]) \text{ for all } \alpha\}$$
$$= \bigcap_{\alpha} T_\alpha^{-1}(S_n[0]),$$

being an intersection of closed sets is closed. By our assumption, for any fixed $x \in B$, Since $\{T_\alpha x\}$ is bounded, $\exists$ $K > 0$ such that $\|T_\alpha x\| \leq K$ for each α. Let n be a +ve integer $\geq K$. Then $\|T_\alpha x\| \leq n$.

Hence, $x \in F_n$. Thus $B = \bigcup_{n=1}^{\infty} F_n$.

Since B is Complete, Baire's Category theorem shows that one of the F_n's say F_{n_0}, has non-empty interior and thus Contains a closed sphere S_0 with Centre x_0 and radius $r_0 > 0$. Hence,

$$T_\alpha(S_0) \subseteq T_\alpha(F_{n_0}).$$

Now, let $y \in T_\alpha(S_0)$. Then $y = T_\alpha x$, $x \in F_{n_0}$.

Hence, $\|y\| = \|T_\alpha x\| \leq n_0$ for all α.

It is clear that $S_0 - x_0$ is a closed sphere with radius r_0 centered on the origin, so $\frac{(S_0 - x_0)}{r_0}$ is the closed unit sphere S, where

$$S_0 - x_0 = \{z - x_0 : z \in S_0\}$$

Since, $x_0 \in S_0$, it is clear that for every $z \in S_0$,

$$\|T_\alpha(z - x_0)\| = \|T_\alpha(z) - T_\alpha(x_0)\| \leq \|T_\alpha(z)\| + \|T_\alpha(x_0)\|$$
$$\leq n_0 + n_0 = 2n_0.$$

Hence, for every $u \in S$, we have

$$\|T_\alpha(u)\| = \left\|T_\alpha\left(\frac{z - x_0}{r_0}\right)\right\| \leq \frac{2n_0}{r_0}, \text{ for every } \alpha.$$

Hence, $\|T_\alpha\| \leq \frac{2n_0}{r_0}$ for every α.

$\therefore \sup_\alpha \|T_\alpha\| < \infty.$

M.U. M.Sc. 95

Q.No> Let A be a subset of a normed linear space E with the property that the set $\{f(x) : x \in A\}$ is bounded for each $f \in E^*$. Then show that A is a bounded subset of E.

Proof:- We know that, $F_x(f) = f(x)$ for each $f \in E^*$. Therefore, the set $\{F_x(f) : x \in A\}$ is bounded for every $f \in E^*$. Since E^* is a Banach space, it follows that the set $\{F_x : x \in A\}$ is bounded. Because $\|F_x\| = \|x\|$, Hence it follows that A is bounded.

This Proves the Corallary.

Let X denote the set of all Polynomials,

$$x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \cdot (a_i \text{ real})$$

where n is not a fixed +ve integer

We can define addition of two Polynomials and Scalar multiplication of a Polynomial to make the set X into a linear space. It can be verified further that X becomes a normed linear space if for

$$x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \in X.$$

We define,

$$\|x\| = \max_j |a_j|.$$

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